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MAXIMAL JORDAN ALGEBRAS OF MATRICES WITH BOUNDED NUMBER OF EIGENVALUES*

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ABSTRACT

We consider maximal Jordan algebras of matrices with bounded number of eigenvalues. Up to simultaneous similarity we list all irreducible algebras of that kind, and we also give a list of some reducible such algebras. We also study automorphisms of Jordan algebras of matrices.

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1. Introduction

The study of linear spaces of matrices having certain non-invertibility properties has a long history. Some interesting lines of investigation were studied by Dieudonné [Di], Flanders [F], Gerstenhaber [G], and Motzkin-Taussky [MT1, MT2]. They consider, respectively, the spaces of singular matrices, matrices of bounded rank, nilpotent matrices, and diagonalizable matrices. Although it is not our intention to give here a complete list of references on the subject, it may be worth pointing out the combinatorial approach to the problem due to Brualdi and Chavey [BC] yielding both results of Flanders' and Gerstenhaber's type. These problems often seek for the maximal possible dimension of a linear space of matrices having certain properties and it turns out that the spaces with this maximal dimension have an interesting structure ensuring the matrices in the space to fulfill the required properties. Some of these problems are quite involved and require deep techniques such as algebraic geometric tools. Additional algebraic structure on the linear space of matrices is often imposed. In a recently studied problem (cf. [At], [OŠ], [LR]) of linear spaces of square matrices whose number of eigenvalues is bounded by a certain number k, the maximal dimension and the corresponding structure of spaces have been determined only for the cases of either k small or k close to the order of the matrices. Since the corresponding problem for (associative) algebras of matrices is trivial, the study of maximal objects with this property within a certain non-associative variety of matrices may shed some light on the problem. Here we give a complete answer to the corresponding question for Jordan algebras of matrices.

Let F be an algebraically closed field of characteristic other than 2. For $n \geq 1$ the (associative) algebra $M_n(F)$ of $n \times n$ matrices with entries in F is a Jordan algebra for the standard **Jordan product** $A \circ B = \frac{1}{2}(AB + BA)$. It is an easy observation that a linear subspace $\mathcal{L} \subset M_n(F)$ is a Jordan subalgebra if and only if $A^2 \in \mathcal{L}$ for all $A \in \mathcal{L}$. It then follows that a Jordan algebra $\mathcal{L}$ of matrices is closed under arbitrary powers, and hence is polynomially closed. A linear subspace $\mathcal{L} \subset M_n(F)$ is called **irreducible** if 0 and F^n are the only common invariant subspaces for all $A \in \mathcal{L}$. We say that a subspace $\mathcal{L}$ of the matrix algebra $M_n(F)$ has property (Pk) if none of the members of $\mathcal{L}$ has more than k distinct eigenvalues and there is a member of $\mathcal{L}$ with exactly k distinct eigenvalues. In this paper we determine all maximal Jordan subalgebras of $M_n(F)$ with property (Pk), $k \geq 2$.

For $k \geq 3$ the only maximal irreducible Jordan algebra with property (Pk) is the full matrix algebra $M_k(F)$. The case $k = 2$ turns out to be exceptional and nontrivial. It exhibits a family of maximal Jordan algebras with an interesting structure. For each $n \geq 1$ we obtain, up to similarity, one maximal irreducible Jordan algebra $\mathcal{J}_n$ with property (P2) in the set of matrices of order 2^n. We study irreducible Jordan algebras with property (Pk) in sections 2 and 3. Before proceeding with the general, not necessarily irreducible case, we investigate the automorphisms of two special types of Jordan algebras. In section 4 we characterize the automorphisms of the Jordan algebras of all strictly upper-triangular matrices and the automorphisms of Jordan algebras of upper-triangular matrices that contain all strictly upper-triangular matrices. In section 5 we characterize the automorphisms of maximal irreducible Jordan algebras with property (Pk). These characterizations of automorphisms are of independent interest. We also use them in section 6 to list all non-isomorphic irreducible Jordan algebras with property (Pk) and some reducible algebras of that kind.

We assume the algebraic closure of the underlying field F so that the spectral projectors and the corresponding nilpotents of elements in $\mathcal{J}$ exist over F and hence are elements of $\mathcal{J}$. It would be enough to assume that all the eigenvalues of elements of $\mathcal{J}$ belong to F.

In the proofs we use methods of matrix theory and linear algebra. Our results could be interpreted also using techniques of Jordan algebras. For the benefit of people familiar with Jordan algebras we do so in the following paragraph. We include a few remarks later in the paper. However, we prefer to work with matrices: The original problem was posed in the language of linear algebra, and, in our results, we exhibit a special form of matrices (e.g., the fractal structure of irreducible Jordan algebras with property (P2), block upper-tringular structure of general maximal Jordan algebras with property (Pk), etc.).

A Jordan algebra $\mathcal{J}$ is called **reduced** if its identity element 1 is a sum of absolutely primitive idempotents. An idempotent e is **absolutely primitive** if $e \neq 0$ and every element in $e\mathcal{J}e$ is of the form $\alpha e + z$, where $\alpha \in F$ and z is nilpotent. Two idempotents e and f are **orthogonal** if $e \circ f = 0$. If $\mathcal{J}$ is a Jordan subalgebra (with identity) in $M_n(F)$ then it is a reduced Jordan algebra [J3, Thm. 4, p. 197]. Moreover, the identity element I is equal to the sum $\sum_{i=1}^{t} e_i$ of orthogonal absolutely primitive idempotents e_i. Following Albert [A1, A2] we call the number t the **degree** of $\mathcal{J}$. (It is called the **capacity** in [J3, p. 158].) If α_j, $j = 1, 2, \ldots, t$, are distinct scalars then $\sum_{i=1}^{t} \alpha_j e_j$ has t distinct eigenvalues and therefore $t \leq k$. On the other hand, if $a \in \mathcal{J}$ has k ditinct eigenvalues then its spectral projectors P_j, $j = 1, 2, \ldots, k$, form an orthogonal set of absolutely primitive idempotents such that $I = \sum_{j=1}^{k} P_j$. This implies that $k \leq t$, and thus

$k = t$. Jordan algebra $\mathcal{J}$ has the Peirce decomposition $\mathcal{J} = \sum_{i\leq j} \mathcal{J}_{ij}$ relative to the idempotents e_i. Here $\mathcal{J}_{ii} = e_i\mathcal{J}e_i$ and $\mathcal{J}_{ij} = e_i\mathcal{J}e_j + e_j\mathcal{J}e_i$ if $i \neq j$ [J3, pp. 197–198]. By the Albert–Jacobson–McCrimmon Theorem [J3, p. 198] $\mathcal{J}_{ii} = Fe_i + \mathcal{N}_{ii}$, where $\mathcal{N}_{ii}$ is the set of nilpotent elements of $\mathcal{J}_{ii}$ and is an ideal in $\mathcal{J}$. If $i \neq j$ then $\mathcal{N}_{ij} = \{x \in \mathcal{J}_{ij}\colon x \circ \mathcal{J}_{ij} = 0\}$ is the set of absolute zero divisors in $\mathcal{J}_{ij}$, it is an ideal in $\mathcal{J}$ such that $x \circ y = 0$ for $x, y \in \mathcal{N}_{ij}$, and $\overline{\mathcal{J}_{ij}} = (\mathcal{J}_{ii} + \mathcal{J}_{jj} + \mathcal{J}_{ij}) \,/\, (\mathcal{N}_{ii} + \mathcal{N}_{jj} + \mathcal{N}_{ij})$ is semisimple (see [J3, Thm. 4, p. 160]). An element x is an **absolute zero divisor** if $x\mathcal{J}x = 0$. In the same way as it is done in the proof of the First Structure Theorem in [J3, pp. 161–162] we show that $\overline{\mathcal{J}} = \sum_{i\leq j} \overline{\mathcal{J}_{ij}}$ is a direct sum of simple algebras, say $\overline{\mathcal{J}} = \sum_{i=1}^{s} \overline{\mathcal{A}}_i$, where $\overline{\mathcal{A}}_i$ are simple Jordan algebras. By the principle of lifting of idempotents [J3, III.7, pp. 148–151] it follows that $\mathcal{J} = \sum_{i=1}^{s} \mathcal{A}_i + \sum_{i\leq j} \mathcal{N}_{ij}$, where $\mathcal{A}_i$ are simple algebras. Now the Albert Structure Theorem [J3, p. 204] gives all the possible finite-dimensional simple Jordan algebras over an algebraically closed field. We show in §2 which of Jordan algebras corresponding to a quadratic form can occur in degree 2 and in §3 that if the degree of $\mathcal{A}_i$ is not 2 then $\mathcal{A}_i$ is isomorphic to $M_{k_i}(F)$ for some integer k_i.

2. Maximal irreducible Jordan algebras with property (P2)

For a block-matrix

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in M_{2n}(F),$$

where each A_i is in $M_n(F)$, we define a blockwise adjoint matrix

$$\widehat{A} = \begin{pmatrix} A_4 & -A_2 \\ -A_3 & A_1 \end{pmatrix}.$$

It is obvious that the operation $\widehat{\ }$ is an involution, i.e., that $\widehat{\widehat{A}} = A$.

For $l = 1, 2, 3, \ldots$ we define subsets $\mathcal{J}_l \subset M_{2^l}(F)$ inductively as follows: $\mathcal{J}_1 = M_2(F)$ and for $l \geq 2$

$$\mathcal{J}_l = \left\{ \begin{pmatrix} \alpha I & A \\ \widehat{A} & \beta I \end{pmatrix} : \alpha, \beta \in F, A \in \mathcal{J}_{l-1} \right\}.$$

It follows from the inductive definition that $\mathcal{J}_l$ is closed under the operation $\widehat{\ }$. For $A \in \mathcal{J}_1$ we write $\tau(A)$ and $d(A)$ for its trace and determinant. For $l \geq 2$ and

$$A = \begin{pmatrix} \alpha I & B \\ \widehat{B} & \beta I \end{pmatrix} \in \mathcal{J}_l$$

we define a linear form $\tau\colon \mathcal{J}_l \to F$ by $\tau(A) = \alpha + \beta$ and we define inductively a quadratic form $d\colon \mathcal{J}_l \to F$ by $d(A) = \alpha\beta - d(B)$. We call τ the **trace form** of $\mathcal{J}_l$. Observe that $A + \widehat{A} = \tau(A)I$ and $A\widehat{A} = d(A)I$.

Note that there is no irreducible Jordan algebra of matrices in $M_n(F)$ with property (P1) for $n \geq 2$. For, if a Jordan algebra $\mathcal{A} \subset \mathcal{M}_n(F)$ satisfies (P1), it is triangularizable by a theorem of Jacobson [R] (see also [J2, Thm. 2, p. 35]), and hence is not irreducible for $n \geq 2$.

PROPOSITION 2.1: *For all l the set $\mathcal{J}_l$ is an irreducible Jordan algebra with property* (P2).

Proof: It is easy to see inductively that $\mathcal{J}_l$ is a linear subspace. Thus, to prove that it is a Jordan algebra it suffices to show that it is closed under squares. A direct computation gives $A^2 - \tau(A)A + d(A)I = 0$ for all $A \in \mathcal{J}_l$. This shows not only that $\mathcal{J}_l$ is closed under squares, but also that each of its members satisfies a quadratic equation. Since

$$\begin{pmatrix} \alpha I & 0 \\ 0 & \beta I \end{pmatrix} \in \mathcal{J}_l$$

has two distinct eigenvalues if $\alpha \neq \beta$, it follows that $\mathcal{J}_l$ has property (P2). It remains to show that it is irreducible. This is clearly true for $l = 1$. Now, for $l \geq 1$ let U be a nontrivial subspace of $F^{2^{l+1}}$ invariant under $\mathcal{J}_{l+1}$. Since the matrices

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

all belong to $\mathcal{J}_{l+1}$, it follows that for each $\begin{pmatrix} x \\ y \end{pmatrix} \in U$ also $\begin{pmatrix} x \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ y \end{pmatrix}$ and $\begin{pmatrix} y \\ x \end{pmatrix}$ belong to U. Thus, the subspace V of F^{2^l}, consisting of all x such that $\begin{pmatrix} x \\ 0 \end{pmatrix}$ is in U, is nontrivial. It is an invariant subspace for all $B \in \mathcal{J}_l$, because

$$\begin{pmatrix} 0 & \widehat{B} \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ Bx \end{pmatrix}.$$

The irreducibility now follows by induction. ∎

PROPOSITION 2.2: *For all l the set $\mathcal{J}_l$ is a maximal Jordan algebra of matrices with property* (P2).

Proof: The proposition is clearly true for $l = 1$. We proceed by induction. Assume that $\widehat{\mathcal{J}}$ is a Jordan algebra with property (P2) and that it contains $\mathcal{J}_{l+1}$.

We define

$$\mathcal{M}=\left\{A_1\colon \text{ there are } A_2,A_3,A_4 \text{ such that } \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\in\widehat{\mathcal{J}}\right\}$$

and

$$\mathcal{N}=\left\{A_2\colon \text{ there are } A_1,A_3,A_4 \text{ such that } \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\in\widehat{\mathcal{J}}\right\}.$$

We will show first that $\mathcal{M}$ has property (P1). Let P be the projection $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. Since $P\in\widehat{\mathcal{J}}$ and $PAP=2P\circ(P\circ A)-P\circ A$ for $A\in\widehat{\mathcal{J}}$ it follows that PAP belongs to $\widehat{\mathcal{J}}$. It is clear now that $\mathcal{M}$ is a Jordan algebra and that $I\in\mathcal{M}$. So, if $\mathcal{M}$ contained a matrix with two distinct eigenvalues, then it would contain a matrix with two distinct nonzero eigenvalues, and consequently, $\widehat{\mathcal{J}}$ would contain a matrix with 3 distinct eigenvalues, which is a contradiction. Thus $\mathcal{M}$ has property (P1). Next, we will prove that $\mathcal{M}$ contains only scalar matrices. We denote by Q the matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. If $X\in\mathcal{M}$ and $\begin{pmatrix} \alpha I & B \\ \widehat{B} & \beta I \end{pmatrix}\in\mathcal{J}_{l+1}$ it follows that

$$\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}=2Q\circ\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}\in\widehat{\mathcal{J}}$$

and

$$\begin{pmatrix} BX+X\widehat{B} & (\alpha+\beta)I \\ (\alpha+\beta)I & XB+\widehat{B}X \end{pmatrix}=2\begin{pmatrix} \alpha I & B \\ \widehat{B} & \beta I \end{pmatrix}\circ\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}.$$

Therefore $BX+X\widehat{B}$ is in $\mathcal{M}$ for all B in $\mathcal{J}_l$. However, $\widehat{B}=\tau(B)I-B$ and so we have that $BX-XB$ is in $\mathcal{M}$. Since $\mathcal{M}$ is a Jordan algebra with property (P1), it is triangularizable by a theorem of Jacobson [R] (see also [J2, Thm. 2, p. 35]). It follows that the intersection U of all kernels of nilpotent matrices from $\mathcal{M}$ is a non-trivial subspace of F^{2^l}. Also, for any nilpotent $N\in\mathcal{M}$ it follows that $NB-BN$ is also a nilpotent matrix in $\mathcal{M}$ (because its trace is zero), so that $NBu=0$ for all $u\in U$. This implies that U is invariant under $\mathcal{J}_l$ and therefore by induction equal to F^{2^n}. So, $\mathcal{M}$ contains only scalar matrices. Finally, choose any matrix of the form

$$A=\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}\in\widehat{\mathcal{J}}.$$

Then, $A\circ Q\in\widehat{\mathcal{J}}$. So $X+Y$ belongs to $\mathcal{M}$ and it is equal to a scalar matrix, say λI. But then $A\circ A\in\widehat{\mathcal{J}}$, and so $XY=X(\lambda I-X)$ also belongs to $\mathcal{M}$ and is equal to a scalar matrix, say μI. It follows that every member of $\mathcal{N}$ satisfies a quadratic equation, namely $X^2=\lambda X-\mu I\in\mathcal{N}$. Therefore, $\mathcal{N}$ is a Jordan algebra. Since it contains $\mathcal{J}_l$, it is equal to it by induction. ∎

THEOREM 2.3: *If $\mathcal{J}$ is a maximal irreducible Jordan algebra of matrices in $M_n(F)$, $n \geq 2$, which has property* (P2) *and contains I, then $n = 2^l$ and $\mathcal{J}$ is simultaneously similar to a $\mathcal{J}_l$.*

Proof: Let $\mathcal{J}$ be an irreducible Jordan algebra containing the identity matrix I. We proceed by induction. If $n = 2$ the theorem is clearly true. Assume that $n \geq 3$. Since $\mathcal{J}$ has property (P2) there is a matrix in $\mathcal{J}$ with two eigenvalues. Consequently, there is a non-trivial idempotent P in $\mathcal{J}$ since $\mathcal{J}$ is closed under arbitrary powers, hence polyimially closed, and since every spectral projection is a polynomial in the matrix. With respect to the block decomposition in which P is of the form $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ we define

$$\mathcal{M} = \left\{ A_1\colon \text{ there are } A_2, A_3, A_4 \text{ such that } \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathcal{J} \right\}$$

and

$$\mathcal{N} = \left\{ A_2\colon \text{ there are } A_1, A_3, A_4 \text{ such that } \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathcal{J} \right\}.$$

Note that for each matrix $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ in $\mathcal{J}$ the matrices

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & A_2 \\ A_3 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix}$$

also belong to $\mathcal{J}$. Arguments similar to those applied in the proof of Proposition 2.2 show that $\mathcal{M}$ satisfies (P1), and by a theorem of Jacobson [R] (see also [J2, Thm. 2, p. 35]) we may therefore assume that $\mathcal{M}$ is in the upper triangular form. Observe that, by symmetry, the set

$$\mathcal{P} = \left\{ A_4\colon \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \hat{\mathcal{J}} \right\}$$

also satisfies (P1).

In order to prove that $\mathcal{M}$ consists of scalar matrices only, we need to show first that the set

$$\mathcal{K} = \left\{ \begin{pmatrix} 0 & A_2 \\ A_3 & 0 \end{pmatrix} \in \hat{\mathcal{J}} \right\}$$

contains an invertible matrix. Assume to the contrary, that all the matrices in $\mathcal{K}$ are singular. Then, the same is true for their squares

$$\begin{pmatrix} 0 & A_2 \\ A_3 & 0 \end{pmatrix}^2 = \begin{pmatrix} A_2 A_3 & 0 \\ 0 & A_3 A_2 \end{pmatrix}.$$

Since A_2A_3 and A_3A_2 each have just one and necessarily the same eigenvalue, this eigenvalue has to be zero. So, they are nilpotent. Denote by $\mathcal{L}$ the set of matrices

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

in $\mathcal{J}$ such that A_1 and A_4 are nilpotent. Since the diagonal blocks of the matrix A^2 are equal to $A_1^2 + A_2A_3$ and $A_4^2 + A_3A_2$, they have trace zero and therefore are nilpotent. This implies that the set $\mathcal{L}$ is a Jordan algebra of trace zero matrices. It follows that for each of them the traces of all its powers are zero. Thus, they are all nilpotent and $\mathcal{L}$ is simultaneously triangularizable. Let U be the (necessarily non-trivial) intersection of all kernels of elements of $\mathcal{L}$. Since with any matrix

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

in $\mathcal{L}$ the matrices

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & A_2 \\ A_3 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix}$$

also belong to $\mathcal{L}$, it follows that PU and $(I - P)U$ are subspaces of U. It is clear that $\mathcal{J}$ is generated by $\mathcal{L}$, P, and I, so it follows that U is invariant under $\mathcal{J}$. Thus, by irreducibility of $\mathcal{J}$ the subspace U is equal to the whole space and $\mathcal{L} = \{0\}$. Again by irreducibility $\mathcal{J}$ contains a matrix

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

with $A_2 \neq 0$ and hence a nonzero matrix A of the form

$$\begin{pmatrix} 0 & A_2 \\ A_3 & 0 \end{pmatrix}.$$

Since $A \in \mathcal{L} = \{0\}$ we get a contradiction. This proves that the set $\mathcal{K}$ contains an invertible matrix A of the above form and that n is even, say $n = 2m$. As above, we see that the square of A has only one eigenvalue. Assume with no loss of generality that it is 1, so that $A_2A_3 = I + N$ for some nilpotent matrix N. Let W be the inverse of the square-root of $I + N$ and observe that this matrix is in $\mathcal{M}$. Then,

$$\begin{pmatrix} 0 & WA_2 \\ A_3W & 0 \end{pmatrix}$$

belongs to $\mathcal{K}$ and its square is equal to I. So, we may assume with no loss of generality that a matrix of the form

$$\begin{pmatrix} 0 & B \\ B^{-1} & 0 \end{pmatrix}$$

belongs to $\mathcal{K}$. By taking a block-diagonal similarity not affecting the upper triangularity of elements of $\mathcal{M}$, we may assume that $\mathcal{K}$ contains the matrix

$$Q = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

It follows that the set $\mathcal{P}$ is equal to $\mathcal{M}$. The intersection U of all the kernels of nilpotent matrices from $\mathcal{M}$ is a non-trivial subspace of F^m. For any matrix

$$\begin{pmatrix} 0 & A_2 \\ A_3 & 0 \end{pmatrix}$$

in $\mathcal{K}$, the matrix $A_2 + A_3$ leaves U invariant and acts as a scalar matrix on it. Since $\mathcal{K}$ contains Q, we may choose the scalar to be 0. Now, for any nilpotent matrix W in $\mathcal{M}$, the matrix

$$\begin{pmatrix} 0 & WA_2 \\ A_3W & 0 \end{pmatrix}$$

belongs to $\mathcal{K}$, so that the matrix $WA_2 + A_3W$ leaves the subspace U invariant and acts like a scalar matrix on it. However, this matrix is equal to the matrix $WA_2 - A_2W$ on U, and since the latter matrix has trace zero, the scalar has to be zero. It follows that A_2 and by symmetry also A_3 both leave the space U invariant, so that $U \oplus U$ is a nonzero invariant subspace for $\mathcal{J}$ and hence $U = F^m$. So, $\mathcal{M}$ contains only scalar matrices. With an argument similar to that in the proof of Proposition 2.2 we now show that every member of $\mathcal{N}$ satisfies a quadratic equation and that it is an irreducible Jordan algebra containing I. By induction it has to be equal to $\mathcal{J}_l$ for some l and therefore $\mathcal{J} = \mathcal{J}_{l+1}$. ∎

3. Maximal irreducible Jordan algebras with property (Pk), $k \geq 3$

First we prove a result for general irreducible Jordan algebras. In particular, it applies to the Jordan algebras $\mathcal{J}_l$.

PROPOSITION 3.1: *If $\mathcal{L} \subset M_n(F)$ is an irreducible Jordan algebra then it is simple.*

Proof: Let $\mathcal{R}$ be the radical of $\mathcal{A}$. It consists of nilpotent elements only [J3, p. 192]. By a theorem of Jacobson [R] (see also [J2, Thm. 2, p. 35]) it is triangularizable. Thus $\mathcal{R}F^n \neq F^n$. Since $A \circ R = \frac{1}{2}(RA + AR)$ for $A \in \mathcal{L}$ and $R \in \mathcal{R}$ it follows that $\mathcal{R}F^n$ is an invariant subspace for $\mathcal{L}$. By irreducibility it is equal to 0, and therefore $\mathcal{R} = 0$ and $\mathcal{L}$ is semisimple. If it was not simple then for each nontrivial ideal $\mathcal{I} \subset \mathcal{L}$ we would have that $\mathcal{I}F^n \neq F^n$ is invariant subspace for $\mathcal{L}$. But this is not possible, and hence $\mathcal{L}$ is simple. ∎

Next we apply the structure theory for simple Jordan algebras developed by Albert and others [A1, A2, J3]. Assume that $\mathcal{A}$ is a maximal irreducible, hence simple, Jordan algebra with property (Pk), $k \geq 3$. Because $\mathcal{A}$ has property (Pk) it follows that the identity matrix I in $\mathcal{A}$ is a sum of k nonzero idempotents P_j in $\mathcal{A}$ such that $P_i \circ P_j = 0$ for $i \neq j$ and it is not a sum of $k+1$ idempotents with these properties. Thus the degree of $\mathcal{A}$ is equal to k. Corollary 2 of [J3, p. 204] implies that the only maximal irreducible algebras of degree k, $k \geq 3$, are $M_k(F)$ and $\mathcal{S}_k = \{A \in M_{2k}\colon A = PA^TP^{-1}\}$, where P is a block diagonal matrix with all diagonal blocks equal to $E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. If α_j, $j = 1, 2, \ldots, k$, are nonzero scalars such that α_j^2 are distinct then the block diagonal matrix

$$\begin{pmatrix} \alpha_1 E & 0 & \cdots & 0 \\ 0 & \alpha_2 E & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_k E \end{pmatrix}$$

is an element of $\mathcal{S}_k$ with $2k$ distinct eigenvalues $\pm\sqrt{-\alpha_j^2}$, $j = 1, 2, \ldots, k$. Therefore $\mathcal{S}_k$ does not have property (Pk). Actually, if F is algebraically closed the degree of $\mathcal{S}_k$ is $2k$ and $\mathcal{S}_k$ does not occur in the classification theorem. This proves the following result.

THEOREM 3.2: *If $\mathcal{A}$ is a maximal irreducible Jordan algebra of matrices in $M_n(F)$, $n \geq 3$, which has property* (Pk), *$k \geq 3$, and containing I, then $n = k$ and $\mathcal{A} = M_k(F)$.*

4. Automorphisms of upper-triangular Jordan algebras

Before we list, up to simultaneous similarity, maximal Jordan algebras with property (Pk) we study isomorphisms of certain special Jordan algebras. We use the results to avoid duplications in the list.

Let us introduce some notation. We write

$$J = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and we denote by E_{ij} the basic matrix with zero entries everywhere except a 1 at the (i,j) entry. Note that J is an involution, i.e., $J^2 = I$. If $\mathcal{U}$ is the Jordan algebra of all upper-triangular matrices in $M_n(F)$ then we call the automorphism $\varphi(A) = JA^TJ$ for $A \in \mathcal{U}$ **the flip**. The proof that φ is an automorphism is straightforward. Finally, if X is a nonempty subset of a vector space V we denote by $\mathcal{L}(X)$ the linear span of X in V.

THEOREM 4.1: *Let $\mathcal{A}$ be Jordan algebra of all strictly upper-triangular matrices in $M_n(F)$ and let $\psi\colon \mathcal{A} \to \mathcal{A}$ be an automorphism. If $n = 3$ then either*

$$\psi(A) = TAT^{-1} \quad \text{for all } A \in \mathcal{A}$$

or

$$\psi(A) = T\varphi(A)T^{-1} \quad \text{for all } A \in \mathcal{A},$$

where T is an invertible upper-tringular matrix and φ is the flip. If $n \geq 4$ then either

$$\psi(A) = TAT^{-1} + \delta(A)E_{1n} \quad \text{for all } A \in \mathcal{A}$$

or

$$\psi(A) = T\varphi(A)T^{-1} + \delta(A)E_{1n} \quad \text{for all } A \in \mathcal{A},$$

where T is an invertible upper-triangular matrix, φ is the flip and $\delta\colon \mathcal{A} \to F$ is a linear map such that $\delta(\mathcal{A}^2) = 0$.

Proof: Let us first show that the above maps are automorphisms of $\mathcal{A}$. If we have $\psi(A) = TAT^{-1} + \delta(A)E_{1n}$ then we obtain

$$\begin{aligned}\psi(A \circ B) &= T(A \circ B)T^{-1} + \delta(A \circ B)E_{1n} = T(A \circ B)T^{-1} \\ &= (TAT^{-1} + \delta(A)E_{1n}) \circ (TAT^{-1} + \delta(A)E_{1n}) = \psi(A) \circ \psi(B).\end{aligned}$$

The second case now follows since φ is an automorphism.

Conversely, we will show that every automorphism is of the above form. Since ψ is a Jordan isomorphism, it preserves powers, so that $\psi(N)$ is a nilpotent of order $n-1$. After an upper-triangular similarity we may assume that $\psi(N) = N$. Now, we introduce the matrix $N_1 = N - E_1$, where $E_1 = E_{12}$. It follows easily that for a in the underlying field the matrix $N_1 + aE_1$ is nilpotent of order no less than $n-2$ and is of order $n-1$ if and only if a is nonzero. So, $\psi(N_1) + a\psi(E_1)$ has the same property. Denote the consecutive entries on the first upper-diagonal of $\psi(N_1)$ by $x_1, x_2, \ldots, x_{n-1}$ and the consecutive entries on the first upper-diagonal of $\psi(E_1)$ by $y_1, y_2, \ldots, y_{n-1}$. The consecutive entries on the $(n-2)$-th upper diagonal of the $(n-2)$-th power of $\psi(N_1) + a\psi(E_1)$ are equal to

$$p = (x_1 + ay_1)(x_2 + ay_2)\cdots(x_{n-2} + ay_{n-2})$$

and

$$q = (x_2 + ay_2)(x_3 + ay_3)\cdots(x_{n-1} + ay_{n-1}),$$

while the $(1, n)$ entry of the $(n-1)$-st power of $\psi(N_1) + a\psi(E_1)$ is equal to

$$r = (x_1 + ay_1)(x_2 + ay_2)\cdots(x_{n-1} + ay_{n-1}).$$

Since ψ is bijective and $\psi(N^{n-1}) = N^{n-1}$ we conclude that for $a = 0$, $r = 0$ and either $p \neq 0$ or $q \neq 0$. Hence, all the entries on the first upper diagonal of $\psi(N_1)$ are nonzero, except for either the first one or the last one. After applying the flip to $\mathcal{A}$, if necessary, we may assume that the zero occurs in the $(1, 2)$ entry. Considering $\psi(N_1) + a\psi(E_1)$ for nonzero a we conclude that $\psi(E_1)$ has a nonzero $(1, 2)$ entry and no other nonzero entries on the first upper diagonal. Apply now the fact that $\psi(N) = N$ to see that $\psi(E_1) = E_1 + X$, and $\psi(N_1) = N_1 - X$, where X is a strictly upper triangular matrix with zeros on the first upper diagonal.

Now, we will show that we may assume with no loss of generality that all the entries of the first row of X are zero. To this end we introduce an upper triangular matrix S of the form $S = I + a_1N + a_2N^2 + \cdots + a_{n-1}N^{n-1}$. Consider the first row of $S(N_1 - X)$. After omitting its first two entries, this row is equal to

$$-\begin{pmatrix} x_{13} & x_{14} & \cdots & x_{1n} \end{pmatrix} + \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-2} \end{pmatrix} \begin{pmatrix} 1 & -x_{24} & \cdots & -x_{2n} \\ 0 & 1 & \cdots & -x_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

It is clear that we can find parameters a_j so that the above row equals zero. It follows that the matrix $S(N_1 - X)S^{-1}$ has zero first row. Observe that this similarity does not change N.

Next, assuming that X has zero first row, we show that it must actually be equal to zero. This will be shown inductively by rows. The fact that E_1^2 is zero implies $0 = (E_1 + X)^2 = E_1 \circ X + X^2$. It follows that the second row of X is equal to zero. Assume that we have already seen that the first $k+1$ rows of X are equal to zero and define $E_{k+1} = E_1 \circ N^k$, so that $\psi(E_{k+1}) = (E_1 + X) \circ N^k = E_{k+1} + X \circ N^k$. From $E_1 \circ E_{k+1} = 0$ we get $0 = (E_1 + X) \circ (E_{k+1} + X \circ N^k) = E_{k+1} \circ X + E_1 \circ (X \circ N^k) + X \circ (X \circ N^k) = 2E_{k+1}X + X \circ (X \circ N^k)$. Only the first term of this last sum may have non-trivial first row equal to the $(k+2)$-th row of X thus proving the desired conclusion. We may now assume that ψ preserves the first row and $\psi(N_1) = N_1$.

The product $E_k A$ is zero except for its first row which is equal to the k-th row of A. Because $E_k A = E_k \circ A$ it follows that $E_k A = \psi(E_k \circ A) = E_k \circ \psi(A) = E_k \psi(A)$. Now,

$$\psi \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & C \end{pmatrix},$$

where C is a strictly upper triangular matrix of order $n-1$. For $A \in \mathcal{A}$ we define a linear map $\chi: \mathcal{A} \to \mathcal{A}$ by

$$\chi(A) = \psi(A) - A.$$

So far our discussion shows that χ has the following properties:

(1) the matrix $\chi(A)$ for $A \in \mathcal{A}$ is zero everywhere except possibly in the first row,
(2) $B\chi(A) = 0$ for all $A, B \in \mathcal{A}$,
(3) $\chi(N^k) = 0$ for $k = 1, 2, \ldots, n-1$,
(4) $\chi(N_1^k) = 0$ for $k = 1, 2, \ldots, n-2$,
(5) $\chi(E_{1j}) = 0$ for $j = 2, 3, \ldots, n$.

Note that if $n = 3$ these properties imply that $\chi = 0$. Therefore ψ has the required form in the case $n = 3$.

In the rest of the proof we assume that $n \geq 4$. First we will establish some further properties of χ:

(6) $\chi(A^2) = \chi(A)A$ for $A \in \mathcal{A}$,
(7) $\chi(A \circ B) = \frac{1}{2}(\chi(A)B + \chi(B)A)$ for $A, B \in \mathcal{A}$,
(8) $\chi(E_{i,i+k}) = \chi(E_{i,i+1})E_{i+1,i+k} + \chi(E_{i+1,i+k})E_{i,i+1}$ for $i = 2, 3, \ldots, n-2$ and $k = 2, 3, \ldots, n-i$.

If we use the definition of χ and property (2) we see that

$$\chi(A^2) = \psi(A^2) - A^2 = \psi(A)^2 - A^2 = (A + \chi(A))^2 - A^2 = \chi(A)A.$$

Thus we have proved (6). Then (7) follows easily from (6) by replacing A by $A + B$, and (8) is a consequence of (7) and of elementary properties of the basic matrices E_{ij}.

Next we study in detail the matrices $\chi(E_{i,i+k})$ for $i = 2, 3, \ldots, n-1$ and $k = 1, 2, \ldots, n-i$. We denote the first row of $\chi(E_{i,i+k})$ by $[h_j^{ik}]_{j=1}^n$. We proceed by the descending induction on k to show that $\chi(E_{i,i+k}) = 0$ for $i = 2, 3, \ldots, n-1$, $k = 2, 3, \ldots, n-1$, and that $h_j^{i1} = 0$ for $i = 2, 3, \ldots, n-1$, $j = 1, 2, \ldots, n-1$.

For $k = n-2$ observe that $E_{2n} = N_1^{n-2}$ and so property (4) implies that $\chi(E_{2n}) = 0$. Our inductive hypothesis is that $\chi(E_{i,i+k'}) = 0$ for $k' > k \geq 1$. Properties (7) and (3) imply that

$$\chi(E_{i,i+k} \circ N) = \frac{1}{2}\left(\chi(E_{i,i+k})N + \chi(N)E_{i,i+k}\right) = \frac{1}{2}\chi(E_{i,i+k})N.$$

On the other hand, since $E_{i,i+k} \circ N = \frac{1}{2}(E_{i,i+k+1} + E_{i-1,i+k})$, the inductive hypothesis implies that

$$\chi(E_{i,i+k} \circ N) = \frac{1}{2}\chi\left(E_{i-1,i+k} + E_{i,i+k+1}\right) = 0.$$

Hence, it follows that $h_j^{ik} = 0$ for $j = 1, 2, \ldots, n-1$.

If $k \geq 2$ then comparison of the right most entries in the first row on both sides of (8) shows that $h_n^{ik} = 0$ and therefore $\chi(E_{i,i+k}) = 0$ for $i < n-k$. Now apply property (4) to get $0 = \chi(N_1^k) = \chi\left(\sum_{i=2}^{n-k} E_{i,i+k}\right) = \chi(E_{n-k,n})$.

The induction process above terminates when $k = 1$. From its results and property (5) we conclude that $\chi(A) = \delta(A)N^{n-1}$, where $\delta\colon \mathcal{A} \to F$ is a linear functional such that $\delta(E_{i,i+k}) = 0$ if $i = 1$ and $k \geq 1$ or $2 \leq i \leq n-2$ and $k \geq 2$. In particular, we see that $\delta(\mathcal{A}^2) = 0$. ∎

Remark 4.2: Two comments are in order. First note that $\chi(A) = \delta(A)E_{1n}$ in the proof of Theorem 4.1 defines an endomorphism of $\mathcal{A}$. The second observation is that in the statement of Theorem 4.1 we may not only assume that $\delta(\mathcal{A}^2) = 0$ but also that $\delta(N) = \delta(N_1) = 0$.

COROLLARY 4.3: *Suppose that $\mathcal{A}$ is a maximal nil Jordan algebra in $M_n(F)$, $n \geq 3$, and $\psi\colon \mathcal{A} \to \mathcal{A}$ is an automorphism. If $n = 3$ then ψ is a similarity or a composition of a similarity and the flip. If $n \geq 4$ then ψ is a composition of a similarity, a trivial perturbation of the identity and, possibly, of the flip. Here by a trivial perturbation of the identity we mean a map $A \mapsto A + \delta(A)e \otimes f$, where δ is a functional on $\mathcal{A}$ such that $\delta(\mathcal{A}^2) = 0$, and e, f are such that $\mathcal{L}(e)$ is the common kernel of $\mathcal{A}$ and $\mathcal{L}(f)$ the common kernel of $\mathcal{A}^*$.*

Proof: A maximal nil Jordan algebra is simultaneously similar to the Jordan algebra of all strictly upper-triangular matrices in $M_n(F)$. The result is then a consequence of Theorem 4.1. ∎

LEMMA 4.4: *Suppose that $\mathcal{B}$ is a Jordan subalgebra of the Jordan algebra of all upper-triangular matrices in $M_n(F)$, $n \geq 3$, such that $\mathcal{B}$ contains the Jordan* *algebra $\mathcal{A}$ of all strictly upper-triangular matrices and that $\psi\colon \mathcal{B} \to \mathcal{B}$ is an automorphism. Then for each pair D, A, where D is a diagonal matrix in $\mathcal{B}$ and A is an element of $\mathcal{A}$ there exists a matrix $A' \in \mathcal{A}$ such that either $\psi(D+A) = D+A'$ or $\psi(D+A) = \varphi(D) + A'$.*

Proof: Since ψ is an automorphism it preserves powers and thus nilpotents. Therefore, it maps $\mathcal{A}$ onto itself. Let $\widehat{\psi}$ be the restriction of ψ to $\mathcal{A}$. By Theorem 4.1 it follows that either $\widehat{\psi}(A) = TAT^{-1} + \delta(A)N^{n-1}$ for all $A \in \mathcal{A}$ or $\widehat{\psi}(A) = T\varphi(A)T^{-1} + \delta(A)N^{n-1}$ for all $A \in \mathcal{A}$. We drop the perturbation part and we denote the obvious extension to $\mathcal{B}$ by σ, i.e., either $\sigma(B) = TBT^{-1}$ for all $B \in \mathcal{B}$ or $\sigma(B) = T\varphi(B)T^{-1}$ for all $B \in \mathcal{B}$. (Note that if $n = 3$ the perturbation part does not occur.) It is easy to check that σ is an automorphism. Let $\chi = \sigma^{-1}\psi$. Then $\chi(A) = A + \alpha N^{n-1}$ for all $A \in \mathcal{A}$ and some scalar α (depending on A). If we take $A \in \mathcal{B}$ then we have

$$\chi(A \circ N^j) = \chi(A) \circ N^j + \alpha_j N^{n-1}$$

for $j = 1, 2, \ldots, n-1$ and some scalars α_j. We choose

$$A = \begin{pmatrix} a_1 & * & \cdots & * \\ 0 & a_2 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

in $\mathcal{B}$ and we write

$$\chi(A) = \begin{pmatrix} b_1 & * & \cdots & * \\ 0 & b_2 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}.$$

The previous relations imply that $a_j + a_{j+l} = b_j + b_{j+l}$ for $j = 1, 2, \ldots, n-l$ and $l = 1, 2, \ldots n-2$. If $n = 3$ then there is no perturbation and these equalities hold for $l = 2$ as well. Now in case of any $n \geq 3$ we conclude that $a_j = b_j$ for all j. Since χ has the property required in the lemma it follows that ψ has it as well. ∎

THEOREM 4.5: *Suppose that $\mathcal{U}$ is the Jordan algebra of all upper-triangular matrices in $M_n(F)$, $n \geq 3$, and that $\psi\colon \mathcal{U} \to \mathcal{U}$ is an automorphism. Then either*

$$\psi(A) = TAT^{-1} \quad \textit{for all } A \in \mathcal{U}$$

or

$$\psi(A) = T\varphi(A)T^{-1} \quad \textit{for all } A \in \mathcal{U},$$

where T is an invertible upper-tringular matrix and φ is the flip.

Proof: We begin as in the proof of Lemma 4.4. Let $\widehat{\psi}$ be the restriction of ψ to $\mathcal{A}$, the Jordan algebra of all strictly upper-triangular matrices. By Theorem 4.1 it follows that either $\widehat{\psi}(A) = TAT^{-1} + \delta(A)E_{1n}$ for all $A \in \mathcal{A}$ or $\widehat{\psi}(A) = T\varphi(A)T^{-1} + \delta(A)E_{1n}$ for all $A \in \mathcal{A}$. We drop the perturbation part and we denote the obvious extension to $\mathcal{U}$ by σ, i.e., either $\sigma(B) = TBT^{-1}$ for all $B \in \mathcal{U}$ or $\sigma(B) = T\varphi(B)T^{-1}$ for all $B \in \mathcal{U}$. In the latter case we replace ψ by $\varphi\psi$ to unify the rest of the proof. By Lemma 4.4 the image $\psi(E_i)$ of the basic matrix $E_i = E_{ii}$, $i = 1, 2, \ldots, n$, has the same diagonal as E_i. Since E_i is an idempotent and ψ preserves squares it follows that the ith row is the only nonzero row of $\psi(E_i)$. Because $\sum_{i=1}^{n} E_i = I$ and $\psi(I) = I$ it follows that $\psi(E_i) = E_i$ and therefore ψ maps diagonal matrices to diagonal matrices.

Now let $\chi = \sigma^{-1}\psi$. For $A \in \mathcal{A}$ we have $\chi(A) = A + \delta'(A)N^{n-1}$, where δ' is a linear functional such that $\delta'(\mathcal{A}^2) = 0$. We will show that $\delta'(\mathcal{A}) = 0$. Choose a matrix

$$A = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

in $\mathcal{A}$. Then for each diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

we have

$$\chi(A \circ D) = \begin{pmatrix} 0 & a_1(d_1+d_2) & 0 & \cdots & b \\ 0 & 0 & a_2(d_2+d_3) & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1}(d_{n-1}+d_n) \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & a_1 & 0 & \cdots & a \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \circ \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

$$= \begin{pmatrix} 0 & a_1(d_1+d_2) & 0 & \cdots & a(d_1+d_n) \\ 0 & 0 & a_2(d_2+d_3) & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1}(d_{n-1}+d_n) \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $a = \delta'(A)$ and $b = \delta'(A \circ D)$. We choose the diagonal entries of D so that $d_i + d_{i+1} = 1$ for $i = 1, 2, \ldots, n-1$ and $d_1 + d_n = 2$. Then we obtain $\chi(A \circ D) = \chi(A)$ and it follows that $2a = a$, or $a = 0$. We conclude that $\chi = id$ and the proof is complete. ∎

5. Automorphisms of maximal irreducible Jordan algebras with property (Pk)

First we consider the case $k = 2$, i.e., Jordan algebras $\mathcal{J}_l$, $l \geq 1$. Jordan algebra $\mathcal{J}_l$ is a simple Jordan algebra by Proposition 3.1. It is then seen directly from its structure that it is a reduced Jordan algebra of degree 2. It is isomorphic to a Jordan algebra of a quadratic form [J3, pp. 13–14 and pp. 202–203]. We now describe the structure precisely. Let $V_l \subset \mathcal{J}_l$ be the set of all matrices in the kernel of the trace form:

$$V_l = \{A \in \mathcal{J}_l\colon\ \tau(A) = 0\}.$$

Observe that d is a nondegenerate quadratic form on V_l and that $\dim V_l = 2l+1$. The associated symmetric bilinear form b on V_l is given by $b(A,B) = \frac{1}{2}\left(d(A+B) - d(A) - d(B)\right)$. The Jordan algebra $\mathcal{J}_l$ is isomorphic to the Jordan algebra $J(V_l) = \mathcal{L}(I) \oplus V_l$ whose product is given by $(\alpha I + A) \circ (\beta I + B) = (\alpha\beta + b(A,B))\, I + \alpha B + \beta A$ for $\alpha, \beta \in F$ and $A, B \in V_l$. The Jordan algebra $J(V_l)$ is special, i.e., it is a Jordan subalgebra of an associative algebra with respect to the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$. By [J3, Thm. 1, p. 261] we know that $J(V_l)$ is a Jordan subalgebra of the Clifford algebra $\mathcal{C}(V_l, d)$. Let $\rho\colon J(V_l) \to \mathcal{C}(V_l, d)$ be the imbedding of Jordan algebras given by $\rho(\alpha I + A) = \alpha I + A + \mathcal{I}$, where $\mathcal{C}(V_l, d) \cong TV_l/\mathcal{I}$ and $\mathcal{I}$ is the ideal of the tensor algebra TV_l generated by elements $A \otimes A - d(A)I$, $A \in V_l$ (see [J3, pp. 260–261]).

It was pointed out by the referee that $\mathcal{C}(V_l, d)$ and $\mathcal{J}_l$ are closely related: the Clifford algebra $\mathcal{C}(V_l, d)$ is the special universal envelope of $\mathcal{J}_l$ [J3, pp. 74–75].

The automorphisms of the Jordan algebra $J(V_l)$ were determined by Jacobson and McCrimmon [JM]. Here we give their matrix version for $\mathcal{J}_l$. We wish to remind the reader that the map $A \to \widehat{A}$ on $\mathcal{J}_l$ is defined in the first paragraph of §2.

LEMMA 5.1: *The linear maps* $\psi_1, \psi_2\colon \mathcal{J}_l \to \mathcal{J}_l$ *defined by* $\psi_1(A) = \widehat{A}$ *and* $\psi_2(A) = A^T$ *are automorphisms of the Jordan algebra* $\mathcal{J}_l$. *Furthermore,*

$$\psi_1(A) = K_l\psi_2(A)K_l^{-1} \quad \text{for all } A \in \mathcal{J}_l,$$

where $K_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and* $K_l = \begin{pmatrix} 0 & K_{l-1} \\ K_{l-1} & 0 \end{pmatrix}$ *for* $l \geq 2$.

Proof: Since $\widehat{(A^2)} = \left(\widehat{A}\right)^2$ and $\left(A^2\right)^T = \left(A^T\right)^2$ it follows that ψ_1 and ψ_2 are Jordan homomorphisms. Clearly they are bijective. To prove the relation between them we proceed by induction on l. Suppose first that $l = 1$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $\mathcal{J}_1$ then

$$K_1A^TK_1^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \widehat{A}.$$

Now assume that

$$K_{l-1}A^TK_{l-1}^{-1} = \widehat{A}$$

for all $A \in \mathcal{J}_{l-1}$ and choose $B = \begin{pmatrix} \alpha I & A \\ \widehat{A} & \beta I \end{pmatrix} \in \mathcal{J}_l$. Then

$$\begin{aligned} K_lB^TK_l^{-1} &= \begin{pmatrix} 0 & K_{l-1} \\ K_{l-1} & 0 \end{pmatrix}\begin{pmatrix} \alpha I & \widehat{A}^T \\ A^T & \beta I \end{pmatrix}\begin{pmatrix} 0 & -K_{l-1} \\ -K_{l-1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \beta I & -K_{l-1}B^TK_{l-1} \\ -K_{l-1}\widehat{B}^TK_{l-1} & \alpha I \end{pmatrix}. \end{aligned}$$

By the induction hypothesis the last matrix above is equal to

$$\begin{pmatrix} \beta I & -\widehat{A} \\ -A & \alpha I \end{pmatrix} = \widehat{B}. \quad \blacksquare$$

THEOREM 5.2: *If* ψ *is an automorphism of the Jordan algebra* $\mathcal{J}_l$ *then either*

$$\psi(A) = SAS^{-1} \quad \text{for all } A \in \mathcal{J}_l$$

or

$$\psi(A) = SK_lA^TK_l^{-1}S^{-1} \quad \text{for all } A \in \mathcal{J}_l,$$

where S *is a product of invertible matrices from* V_l, *and* K_l *is given in Lemma 5.1.*

Proof: The automorphism ψ induces an automorphism $\psi\colon J(V_l) \to J(V_l)$. It maps I to I and we want to show that it maps V_l onto itself. Assume that

$A \in V_l$ is nonzero and that $\psi(A) = \beta I + B$. Since ψ is one-to-one it follows that $B \neq 0$. Then we have that $d(A)I = \psi(d(A)I) = \psi(A^2) = \psi(A)^2 = (\beta I + B)^2 = (\beta^2 + d(B))\, I + 2\beta B$. It follows that $\beta = 0$ and that $d(A) = d(\psi(A))$. Now we see that ψ induces an automorphism of the quadratic space (V_l, d) and, moreover, an automorphism of the Clifford algebra $\mathcal{C}(V_l, d)$. Since $\dim V_l$ is an odd number and F is algebraically closed it follows that $\mathcal{C}(V_l, d)$ is isomorphic, say via an isomorphism χ, to the direct sum $M_{2^l}(F) \oplus M_{2^l}(F)$ (see, e.g., [D, Thm. VIII.8 and Cor. VIII.11]). From the construction of the isomorphism χ in the proof of [D, Thm. VIII.8] we see that each copy of $M_{2^l}(F)$ contains a copy of the Jordan algebra $J(V_l)$. So we may assume that $\rho\colon \mathcal{J}_l \to M_{2^l}(F) \subset \mathcal{C}(V_l, d)$. Each automorphism of a quadratic space is either of the form

$$\psi(A) = SAS^{-1} \quad \text{for all } A \in V_l$$

or of the form

$$\psi(A) = -SAS^{-1} \quad \text{for all } A \in V_l,$$

where S is a product of invertible matrices from V_l and the product SAS^{-1} is taken in the Clifford algebra $\mathcal{C}(V_l, d)$ (see [D, p. 352]). Note that $\widehat{A} = -A$ for $A \in V_l$. Then Lemma 5.1 implies that $K_l\,(\alpha I + A)^T K_l^{-1} = \alpha I - A$ for $\alpha \in F$ and $A \in V_l$. The result now follows easily. ∎

The following theorem resolves the case $k \geq 3$. It is a result of Ancochea [An] (see also Jacobson [J1, Thm. 2]). Recall that by the Skolem-Noether Theorem (see, e.g., [C, Cor. 7.1.8]) every automorphism of $M_k(F)$ is inner and that the transposition is an antiautomorphism of $M_k(F)$.

THEOREM 5.3: *If $\psi\colon M_k(F) \to M_k(F)$ is an automorphism of Jordan algebra then either*

$$\psi(A) = SAS^{-1} \quad \textit{for all } A \in M_k(F)$$

or

$$\psi(A) = SA^TS^{-1} \quad \textit{for all } A \in M_k(F),$$

where S is an invertible matrix.

6. Maximal Jordan algebras with property (Pk)

If a Jordan algebra $\mathcal{A} \subset M_n(F)$ with property (Pk) is not irreducible then there exists a maximal nontrivial invariant subspace $U \subset F^n$ for $\mathcal{A}$ and complementary subspace $W \subset F^n$ such that $F^n = U \oplus W$ and each element in $\mathcal{A}$ is of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

Let P_W: $F^n \to W$ be the projection on W along U. Then it is easy to see that $\mathcal{A}_U = \{A|_U\colon A \in \mathcal{A}\}$ and $\mathcal{A}_W = \{P_W A|_W\colon A \in \mathcal{A}\}$ are Jordan algebras with property (Pr), $r \leq k$. If either of them is not irreducible we continue the procedure until we have a decomposition $F^n = V_1 \oplus V_2 \oplus \cdots \oplus V_t$ such that all the Jordan algebras $\mathcal{A}_i = \mathcal{A}_{V_i}$ are irreducible. We call the Jordan algebras $\mathcal{A}_i$ irreducible components of $\mathcal{A}$.

Suppose that there exists an integer l such that all the irreducible components $\mathcal{A}_i$ are isomorphic to $\mathcal{J}_l$. Let H be a subset of $\{2,3,\ldots,t\}$. Then we denote by $\mathcal{A}_{2,l,H}$ the Jordan algebra of all block upper-triangular matrices with the j-th diagonal block equal to B if $j \notin H$ and B^T if $j \in H$ for some $B \in \mathcal{J}_l$.

THEOREM 6.1: *If $\mathcal{A}$ is a maximal Jordan algebra with property* (P2) *such that each of the irreducible components $\mathcal{A}_i$, $i = 1,2,\ldots,t$, has property* (P2) *then there exists $l \in \mathbb{N}$ and a subset $H \subset \{2,3,\ldots,t\}$ such that $\mathcal{A}$ is simultaneously similar to the Jordan algebra $\mathcal{A}_{2,l,H}$.*

Proof: If $t = 1$ then the theorem follows by Theorem 2.3. Suppose now that $t \geq 2$. By irreducibility it follows that each $\mathcal{A}_i$ is isomorphic to Jordan algebra $\mathcal{J}_l$ for some l. Let $U = V_1 \oplus V_j$ for some $j \neq 1$. For each element $A \in \mathcal{A}$ we write

$$P_U A|_U = \begin{pmatrix} A_1 & B \\ 0 & A_j \end{pmatrix}.$$

Suppose that $A \in \mathcal{A}$ is such that $A_1 = 0$. We want to show that then $A_j = 0$. We define

$$\mathcal{N} = \left\{ D \in \mathcal{A}_j\colon \text{ there is } A \in \mathcal{A} \text{ such that } P_U A|_U = \begin{pmatrix} 0 & * \\ 0 & D \end{pmatrix} \right\}.$$

Observe that $CA_j + A_jC \in \mathcal{N}$ for all $A_j \in \mathcal{A}_j$ and $C \in \mathcal{N}$. Then it follows that $\mathcal{N}$ must be a Jordan algebra with property (P1). So the intersection K of the kernels of all the nilpotent elements in $\mathcal{N}$ is a nonzero invariant subspace for $\mathcal{A}_j$. By the irreducibility of $\mathcal{A}_j$ it follows that $K = V_j$ and thus $\mathcal{N}$ consists of scalar matrices only. Since $\mathcal{A}_1$ has property (P2) it follows that $\mathcal{N} = 0$. Thus the map ϕ_j: $\mathcal{A}_1 \to \mathcal{A}_j$ given by $\phi_j(A_1) = A_j$ for $A_1 \in \mathcal{A}_1$ such that there is an $A \in \mathcal{A}$ with

$$P_U A|_U = \begin{pmatrix} A_1 & B \\ 0 & A_j \end{pmatrix}$$

is well defined. If we exchange the roles of $\mathcal{A}_1$ and $\mathcal{A}_j$ above we see that ϕ_j is injective. It is also surjective and a Jordan algebra homomorphism, thus it is an isomorphism. Since $\mathcal{A}$ is maximal with property (P2) also $\mathcal{A}_i$ for $i =$

$1, 2, \ldots, t$ is maximal with property (P2). It is irreducible and by Theorem 2.3 it is simultaneously similar to $\mathcal{J}_l$ for some l. Assume that all $\mathcal{A}_i$ are already equal to $\mathcal{J}_l$. By maximality of $\mathcal{A}$ the blocks above the block-diagonal are arbitrary. By Theorem 5.2 it follows that for each $j = 2, 3, \ldots, t$, either

$$\phi_j(A) = SAS^{-1} \quad \text{for all } A \in \mathcal{A}_1$$

or

$$\phi_j(A) = SA^TS^{-1} \quad \text{for all } A \in \mathcal{A}_1,$$

for some invertible matrix S depending on j. Now let H be the set of all j such that ϕ_j involves transposition. It follows then that $\mathcal{A}$ is simultaneously similar to $\mathcal{A}_{2,l,H}$. ∎

Suppose that all the irreducible components $\mathcal{A}_i$ are isomorphic to $M_k(F)$, $k \geq 3$. Let H be a subset of $\{2, 3, \ldots, t\}$. Then we denote by $\mathcal{A}_{k,H}$ the Jordan algebra of all block upper-triangular matrices with the j-th diagonal block equal to B if $j \notin H$ and B^T if $j \in H$ for some $B \in M_k(F)$.

The proof of the following result uses Theorems 3.2 and 5.3 and is similar to the proof of Theorem 6.1.

THEOREM 6.2: *If $\mathcal{A}$ is a maximal Jordan algebra with property* (Pk), $k \geq 3$, *such that all of the irreducible components of $\mathcal{A}$ have property* (Pk) *then all the irreducible components are isomorphic to $M_k(F)$ and $\mathcal{A}$ is simultaneously similar to a Jordan algebra $\mathcal{A}_{k,H}$ for some subset $H \subset \{2, 3, \ldots, t\}$.*

The following result follows directly from Theorem 4.5.

THEOREM 6.3: *A Jordan algebra $\mathcal{A}$ with property* (Pk) *with all its simple parts of dimension* 1 *is isomorphic to one of the Jordan algebras of upper-triangular matrices with k sets of linked entries on the diagonal and arbitrary entries above the diagonal. Moreover, two such Jordan algebras are isomorphic if and only if they can be obtained from each other by applying the flip.*

If at least one of the irreducible components of $\mathcal{A}$ has property (Pj) with $j < k$ then the irreducible components have property (Pr_i) with $r_i < k$ and k is equal to the sum of all the values of r_i that appear in blocks that are not linked. The Jordan algebra $\mathcal{A}$ is simultaneously similar to a Jordan algebra $\widehat{\mathcal{A}}$ in block upper-triangular form, where for those r_i that are different from 2 the diagonal blocks corresponding to irreducible components with property (Pr_i) are equal to the full matrix algebra $M_{r_i}(F)$. The diagonal blocks corresponding to the irreducible components with $r_i = 2$ are equal to $\mathcal{J}_l$ for some l.

In case distinct r_i's do occur we do not give a complete list of non-isomorphic Jordan algebras with property (Pk). The generalization of Theorem 4.5 to the block upper-triangular case would give a result similar to Theorem 6.3, and hence a complete classification of Jordan algebras with property (Pk). However, the calculations required for such a generalization appear to be technically quite formidable.

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